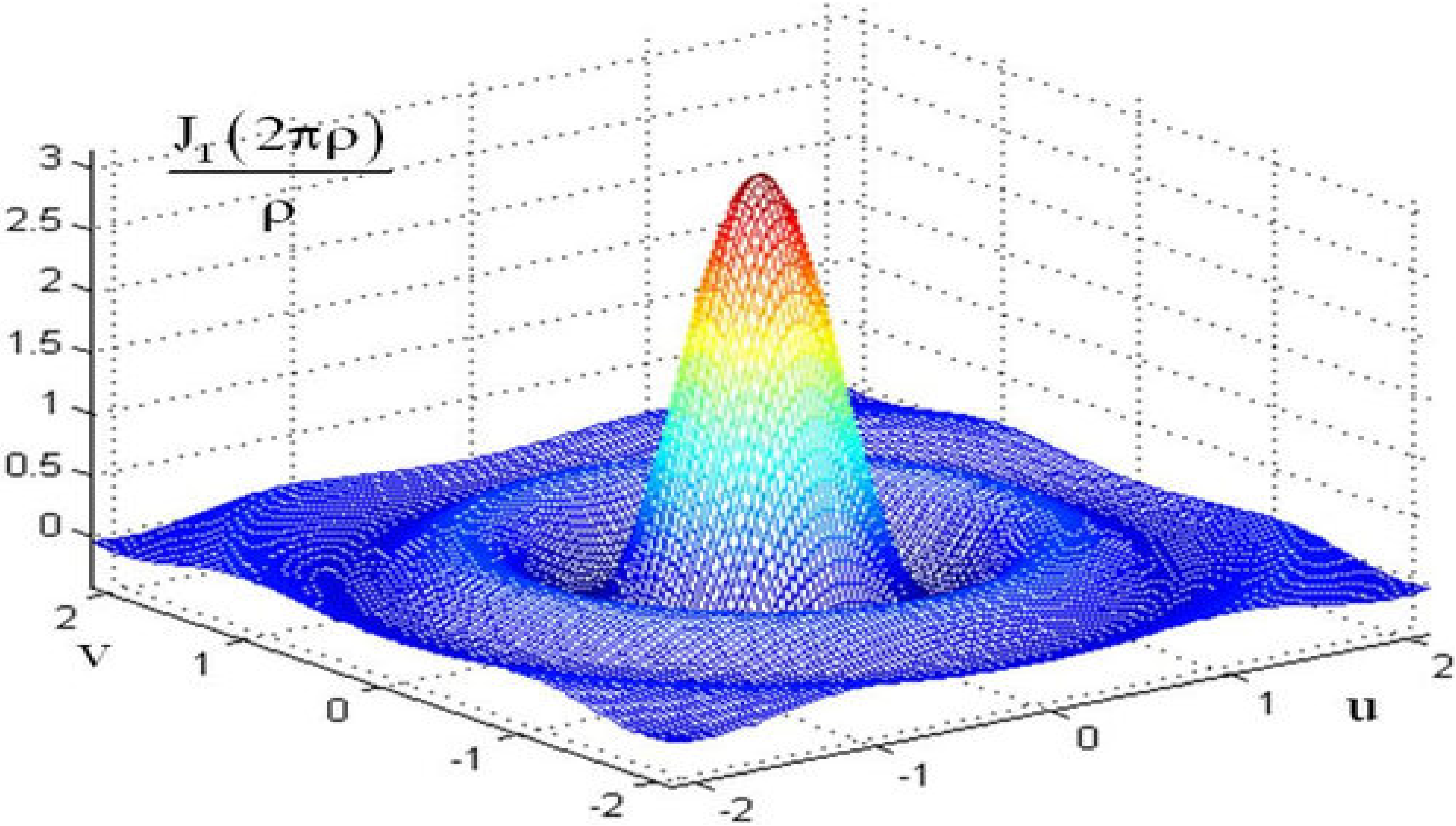
Fourier transforms



Kavilan Moodley, UKZN

Slides taken from R Srianand's lectures: SKA School, Dec 2013

Fourier transforms in astronomy

- Time series analysis
- Imaging of sources particularly in radio
- Power-spectrum studies.

DEFINITION OF FT:

The Fourier transform:

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) e^{-i\omega t} dt$$

The inverse Fourier transform:

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(\omega) e^{i\omega t} d\omega$$

This integration with the basic function is like projection of a function in "t" domain to "ω" domain.

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t)[cos(\omega t) + isin(\omega t)]dt$$

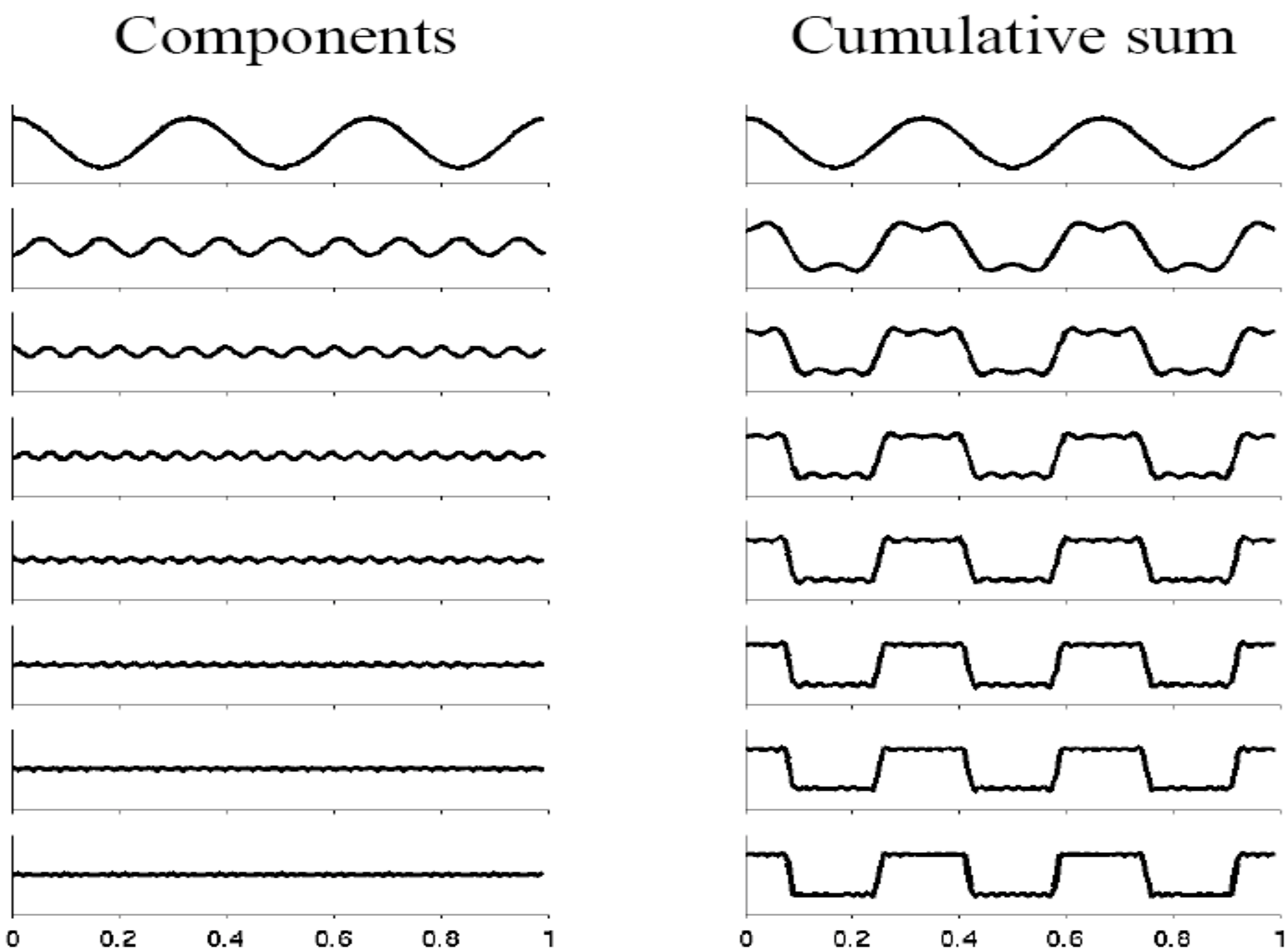
$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) cos(\omega t) dt + \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) sin(\omega t) dt$$

Both real and imaginary components are needed to capture the amplitude ($|F(\omega)|$) and phase ($tan^{-1}(Im(F(\omega))/Re(F(\omega)))$) at every frequency.

The power at any frequency ω is $|F(\omega)|^2$. ⟶ Power Spectrum

$$F(\omega) = A_\omega e^{i\Phi_\omega}$$

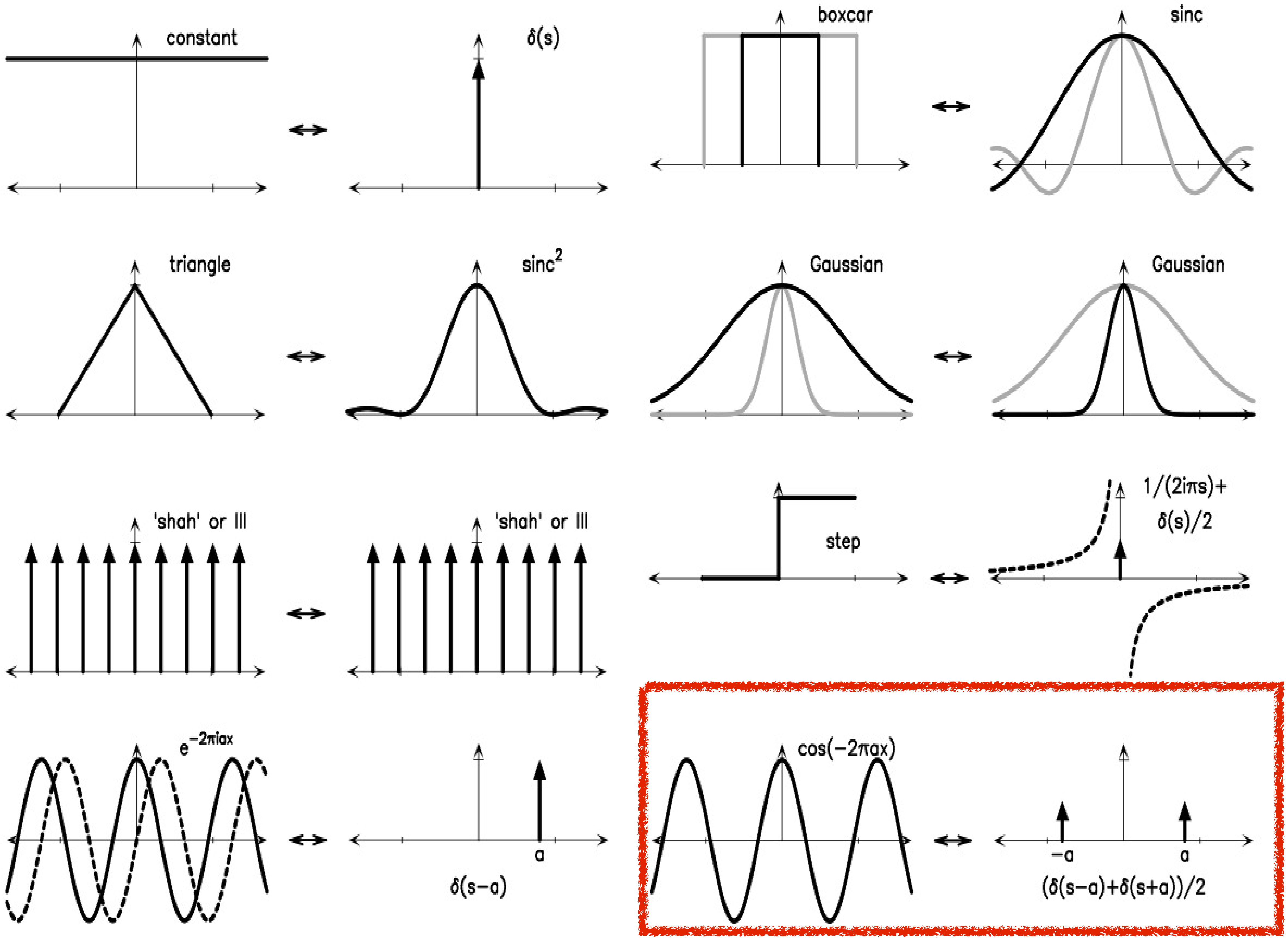
DECOMPOSITION OF A SQUARE WAVE:



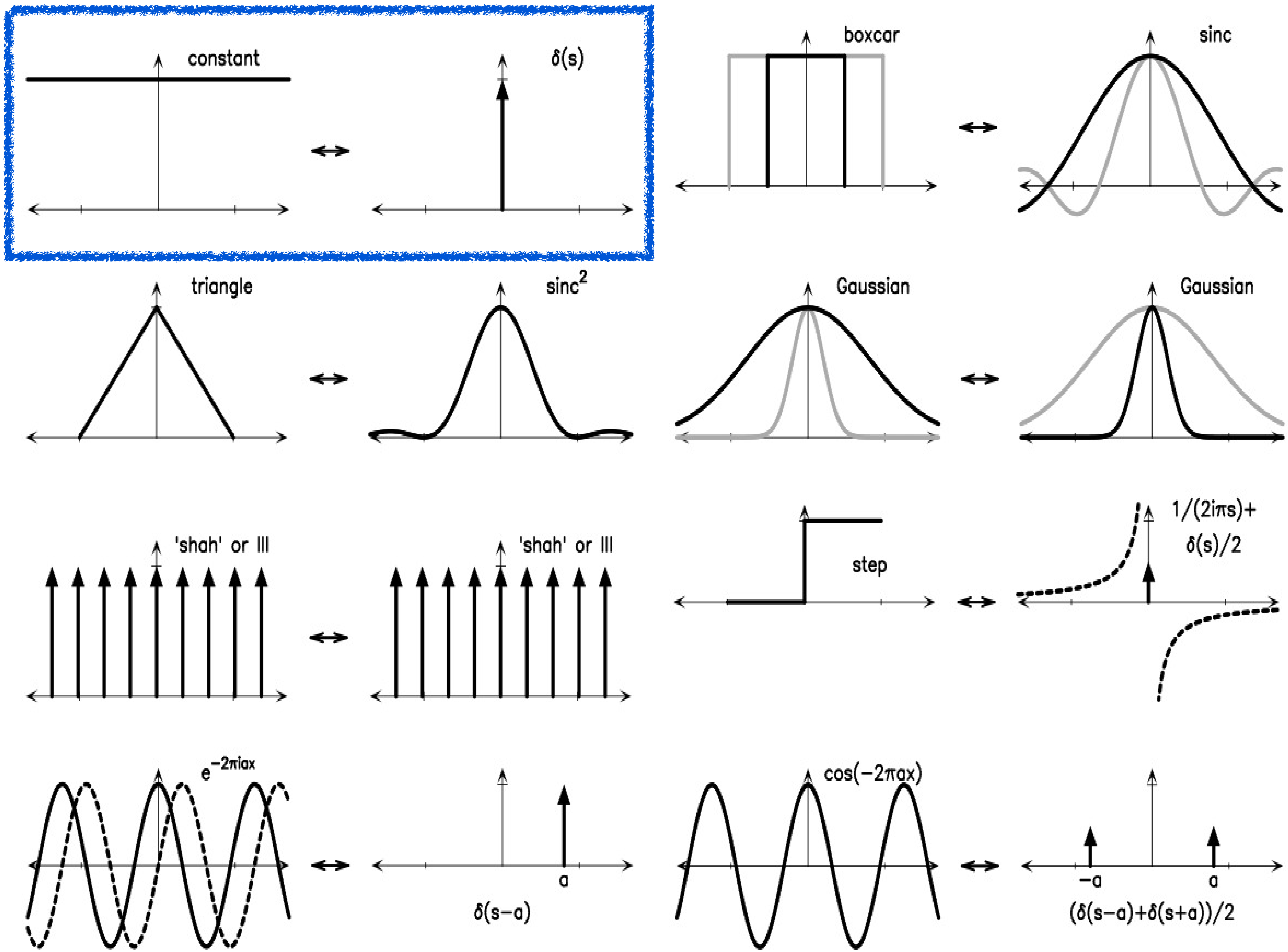
Different harmonics

$f(t) = \Sigma\, a_n\, f_n(t)$

SOME STANDARD FT PAIRS:



Some standard FT pairs:



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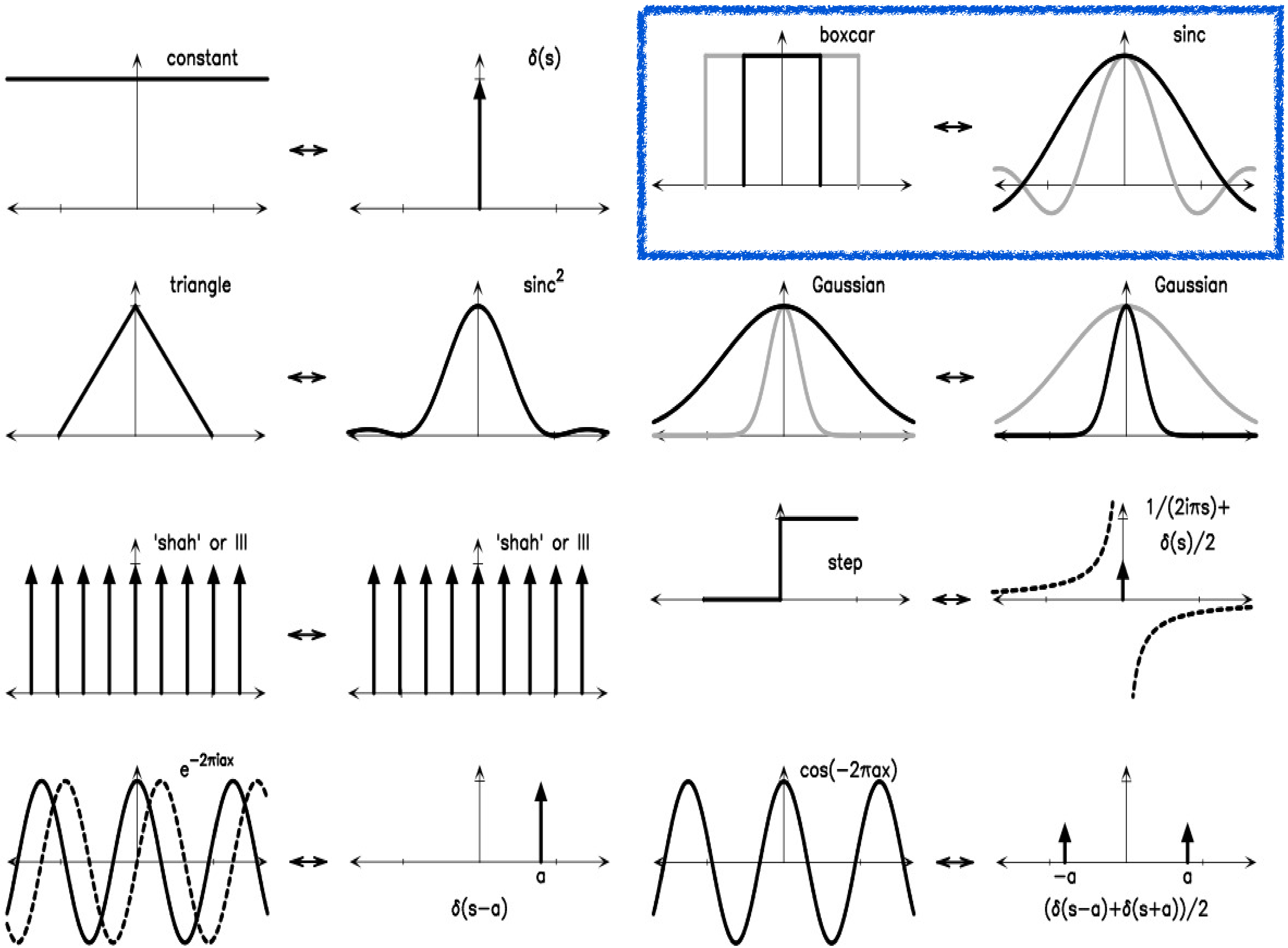

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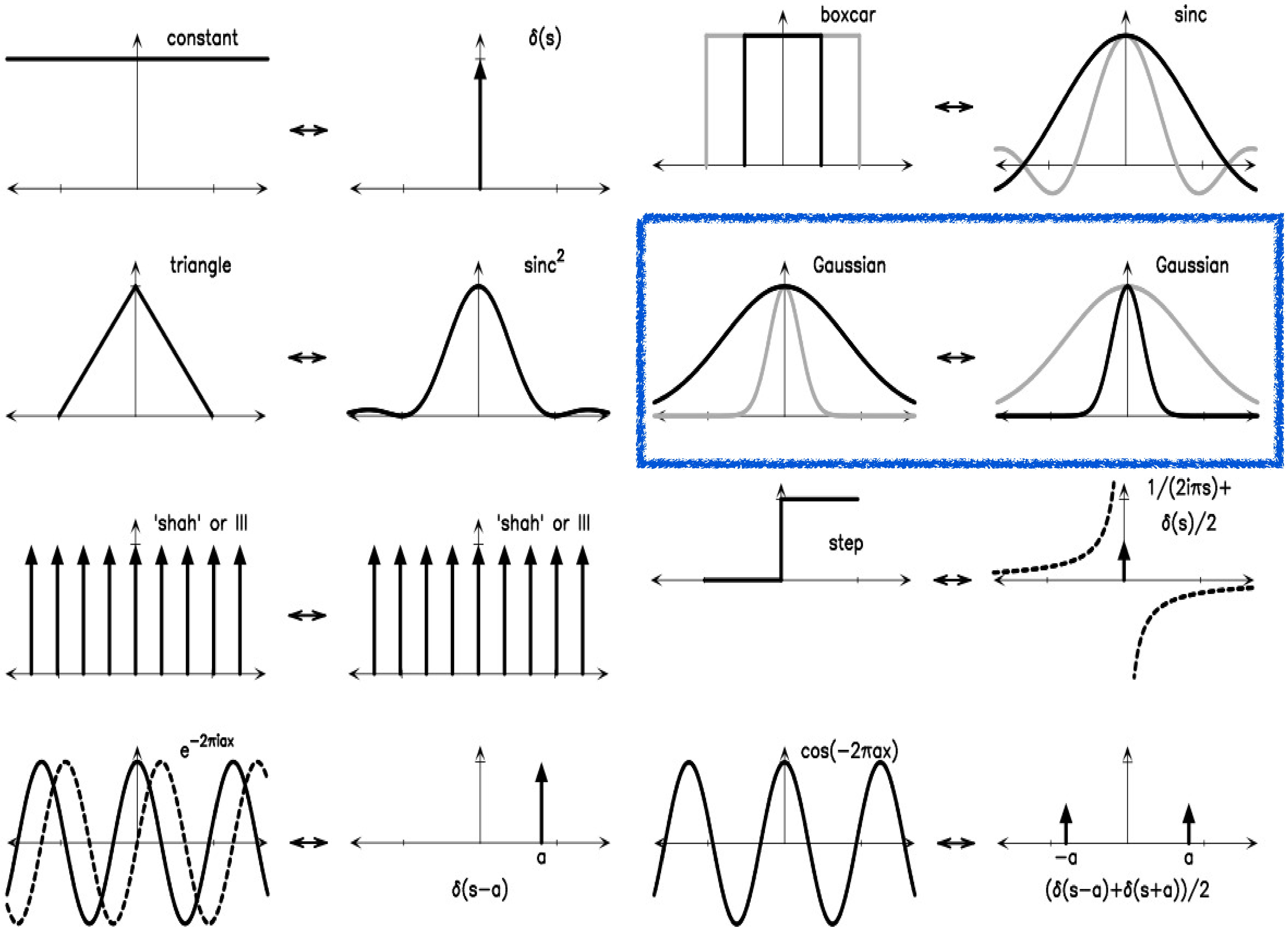
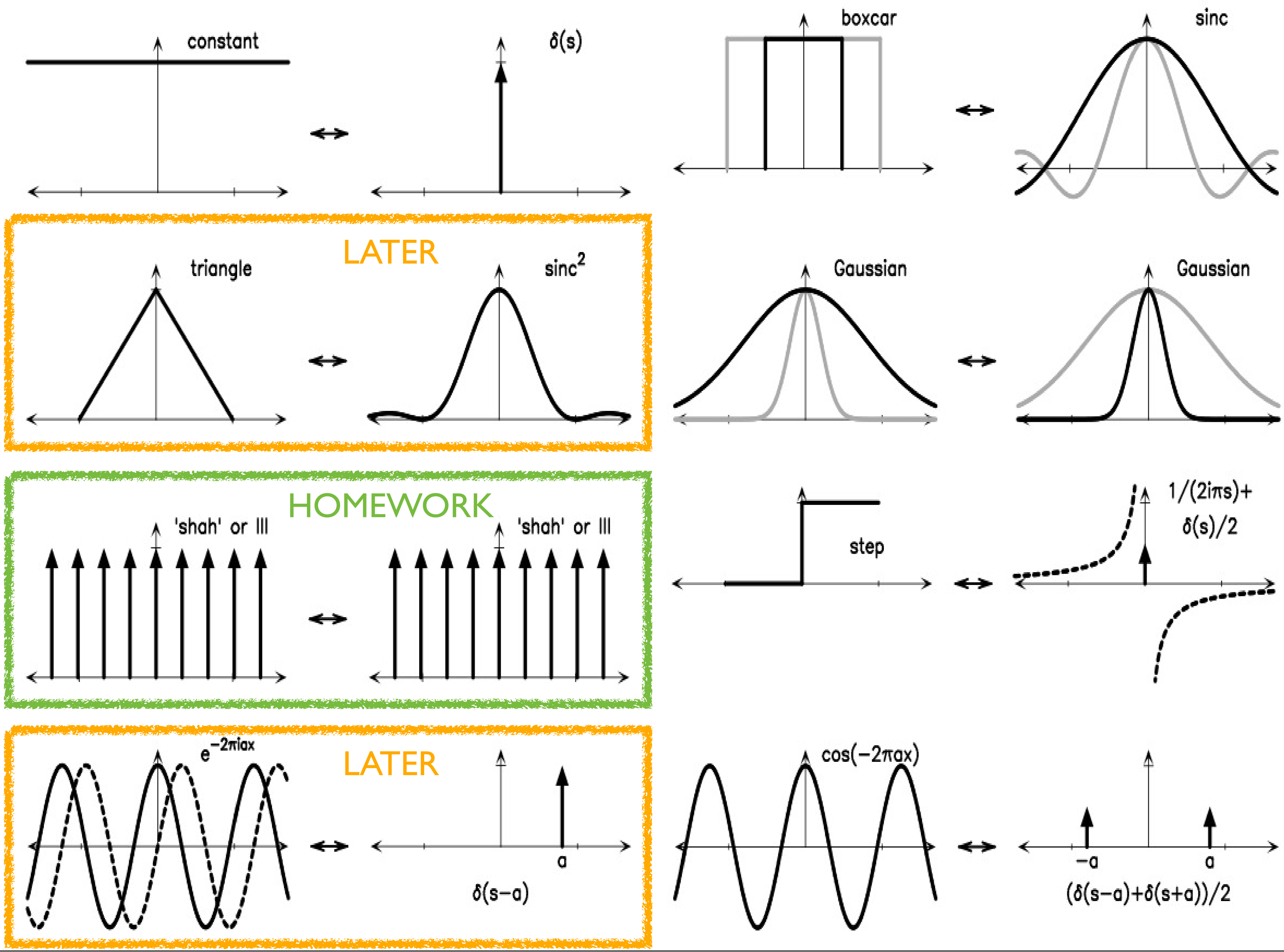


SOME STANDARD FT PAIRS:



PROPERTIES OF FT: LINEARITY PROPERTIES

(1) If f(t) and g(t) are functions with transforms F(ω) and G(ω), respectively, then

Fourier Transform Operator

$$\mathcal{F}[f(t) + g(t)] = F(\omega) + G(\omega) \quad (1)$$

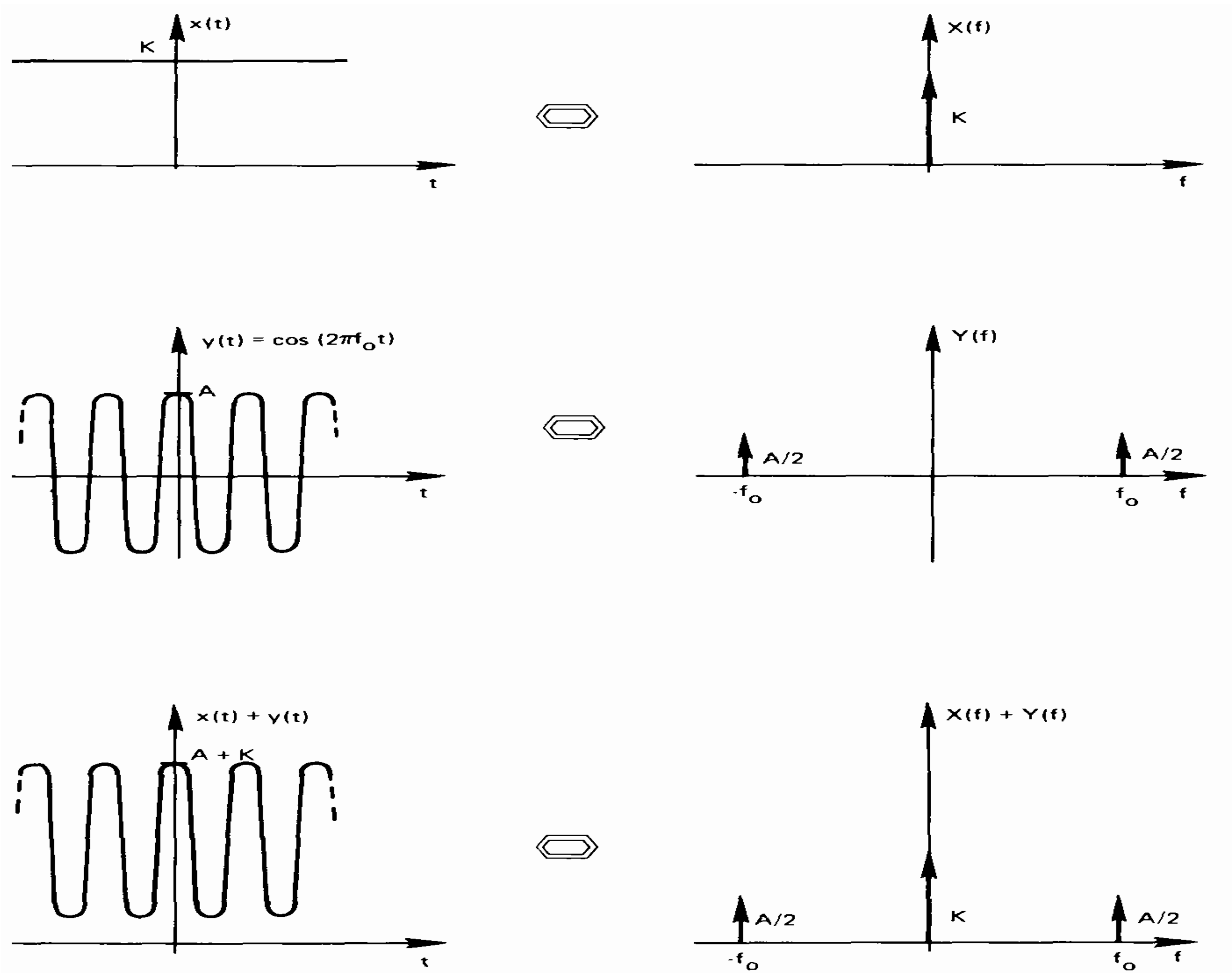
If we add two functions then the Fourier Transform of the resulting function is simply the sum of the individual fourier transforms.

(2) if k is any constant,

$$\mathcal{F}[kf(t)] = kF(\omega) \quad (2)$$

if we multiply a function by a constant then we must multiply its FT by the same constant.

LINEARITY: EXAMPLE



PROPERTIES OF FT: SCALING

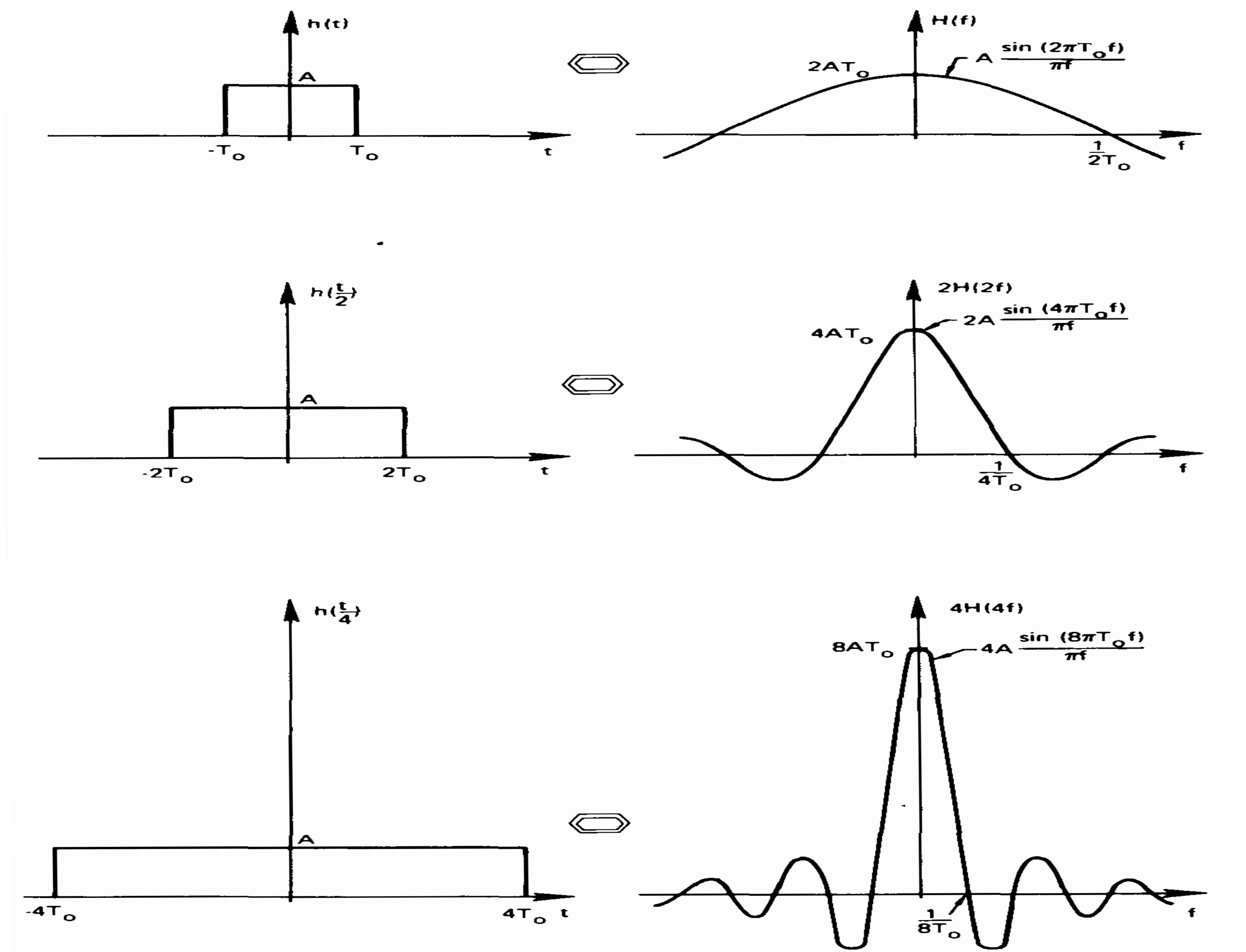
Time-scaling,

$$\mathcal{F}[f(kt)] = \frac{1}{|k|}F(\frac{\omega}{k})$$

Frequency-scaling

$$\mathcal{F}[\frac{1}{|k|}f(\frac{t}{k})] = F(\omega k)$$

TIME-SCALING: EXAMPLE



Frequency-scaling: Example

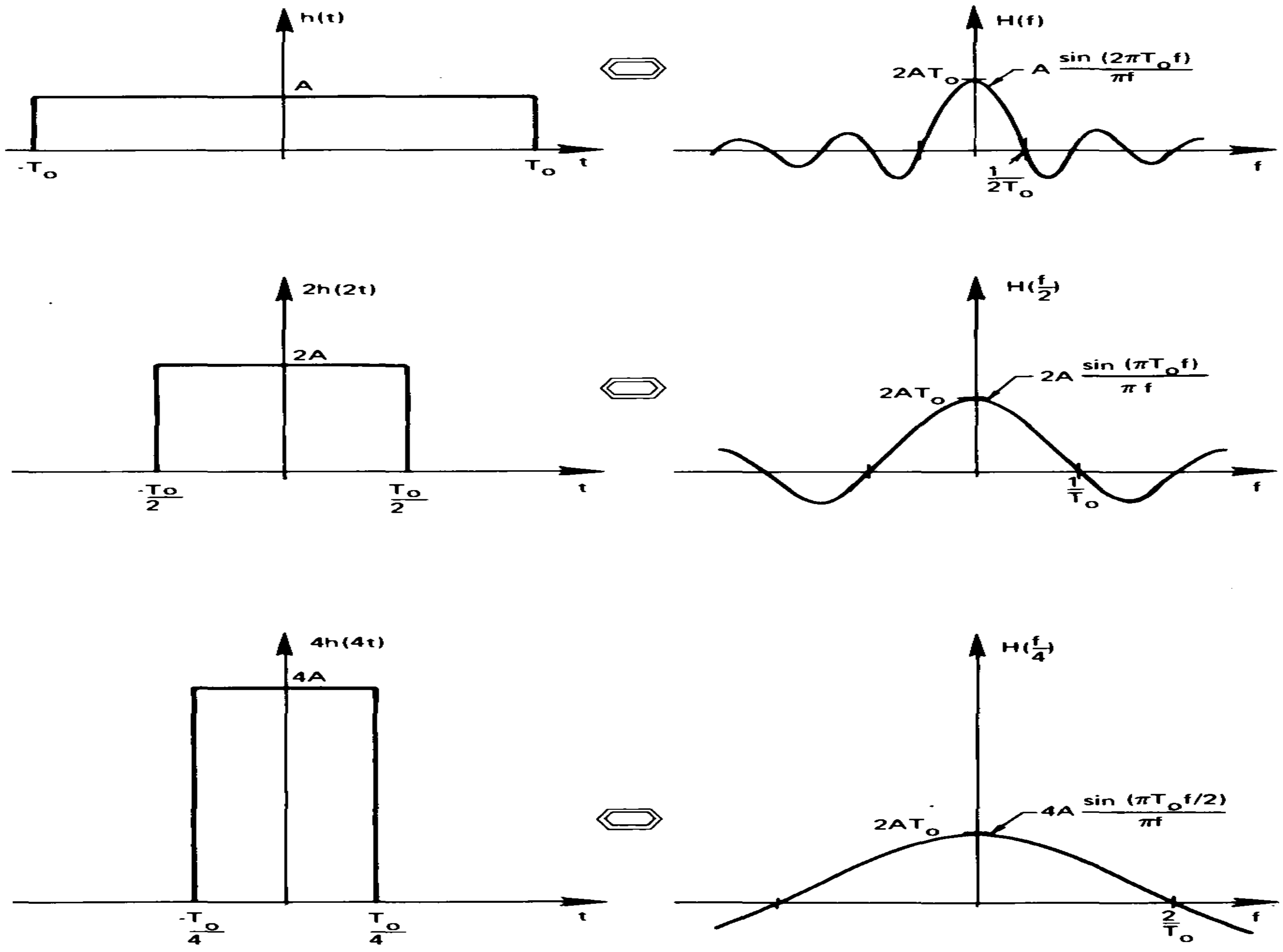


Figure 3-3. Frequency scaling property

Properties of FT: Shift Properties

(i) Time shifting property:

$$\mathcal{F}[f(t - t_0)] = e^{-i\omega t_0} F(\omega) \tag{3}$$

(ii) Frequency shifting property:

$$\mathcal{F}[e^{-i\omega t_0} f(t)] = F(\omega - \omega_0) \tag{4}$$

Here, t_0 and ω_0 are constants. Shifting (or translating) a function in one domain corresponds to a multiplication by a complex exponential function in the other domain.

Example: shifted Gaussian in 1-d and 2-d

PROPERTIES OF FT: SHIFT PROPERTIES

$$\mathcal{F}[f(t-t_0)] = \int_{-\infty}^{+\infty} f(t-t_0)e^{-i\omega t}dt \quad (5)$$

$$= \int_{-\infty}^{+\infty} f(\tau)e^{-i\omega(\tau+t_0)}d\tau \quad (6)$$

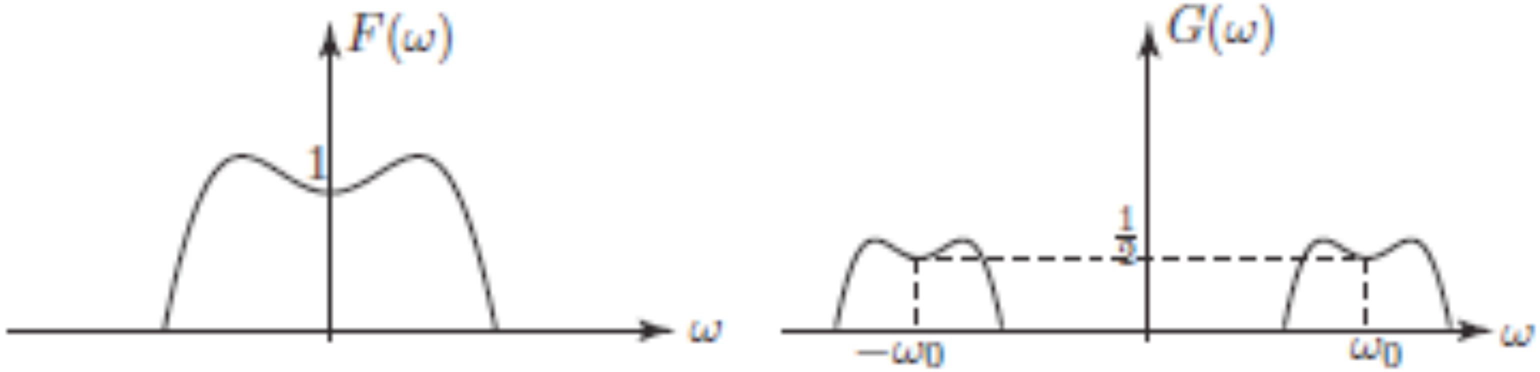
$$= e^{-i\omega(t_0)} \int_{-\infty}^{+\infty} f(\tau)e^{-i\omega(\tau)}d\tau \quad (7)$$

$$= e^{-i\omega t_0}F(\omega) \quad (8)$$

Properties of FT: Shift Properties

Find the FT of a modulated wave, $g(t) = f(t)\ cos(\omega_0 t)$.
We have,

$$
\begin{aligned}
g(t) &= f(t)\left(\frac{e^{i\omega_0 t} + e^{-i\omega_0 t}}{2}\right) &\quad (9)\\
&= \frac{1}{2} f(t) e^{i\omega_0 t} + \frac{1}{2} f(t) e^{-i\omega_0 t} &\quad (10)\\
\mathcal{F}[g(t)] &= \frac{1}{2} F(\omega - \omega_0) + \frac{1}{2} F(\omega + \omega_0) &\quad (11)
\end{aligned}
$$



TIME-SHIFTING: EXAMPLE

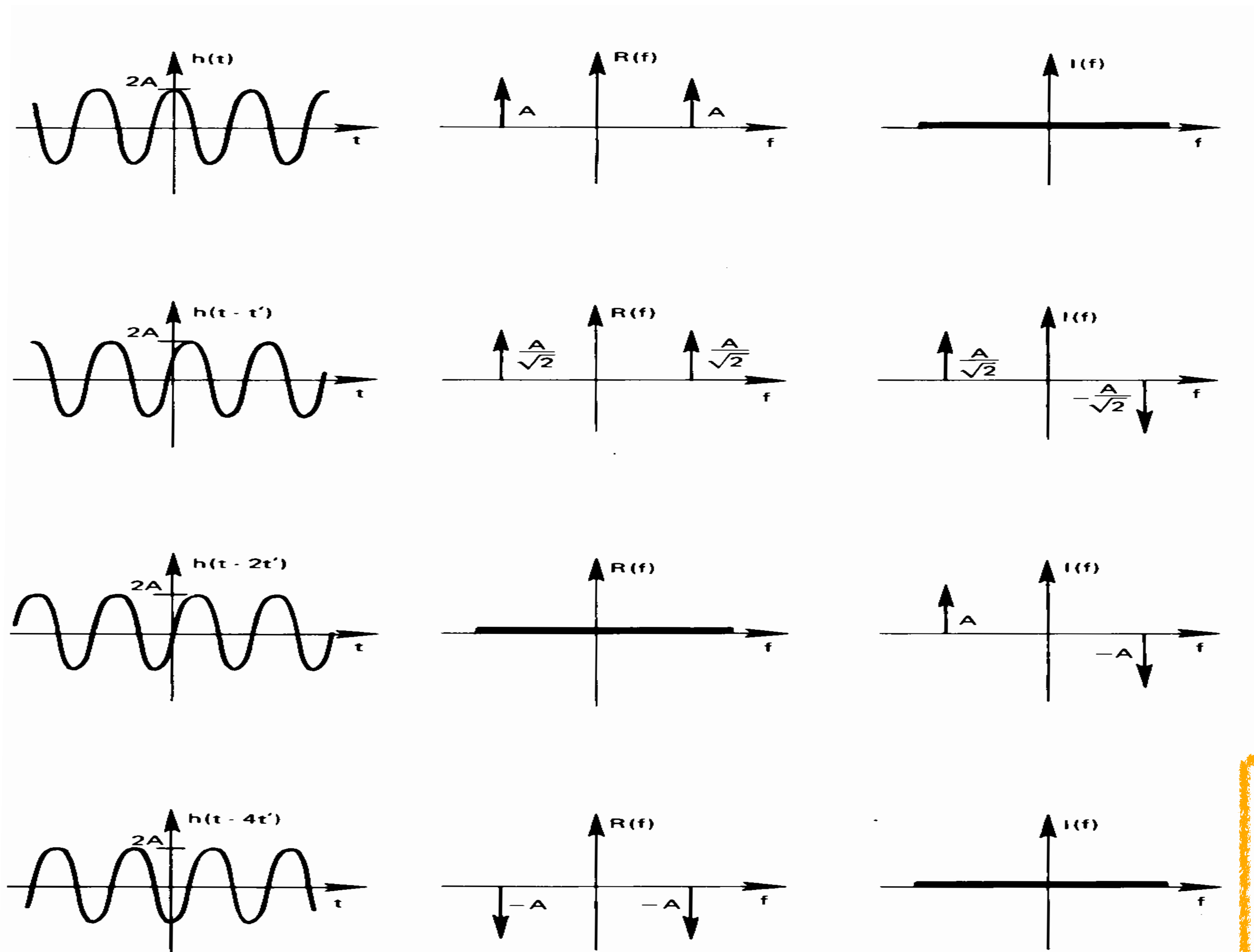
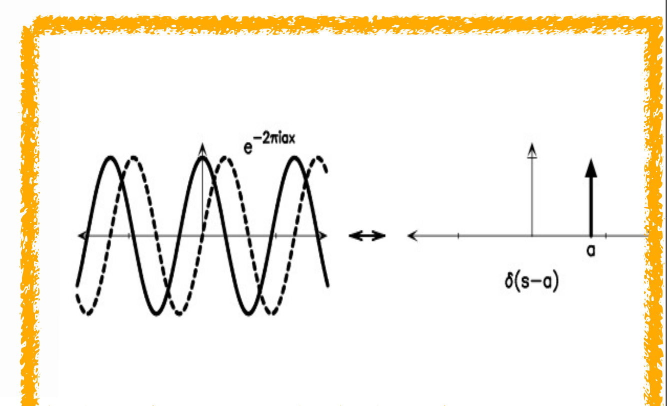
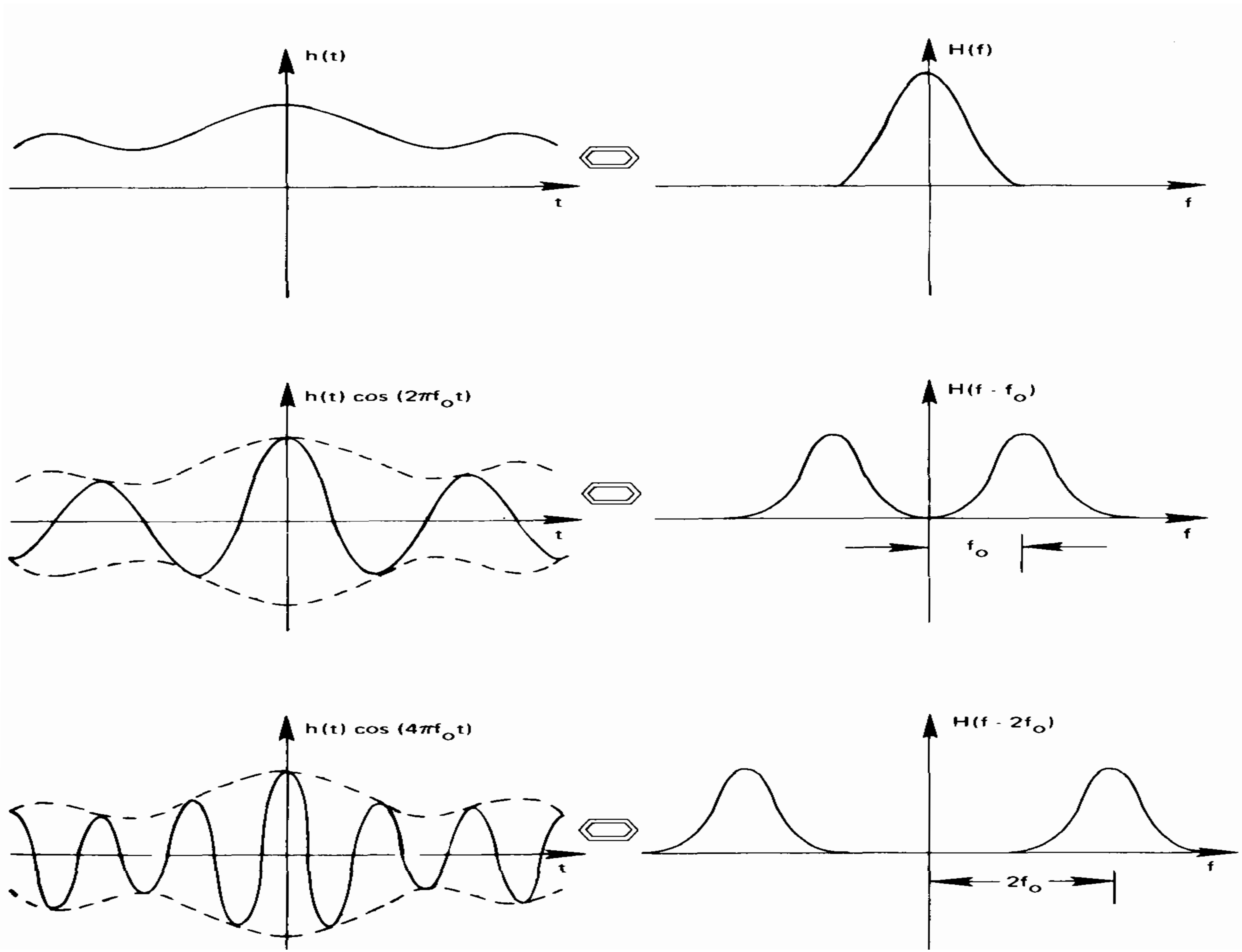


Figure 3-4. Time shifting property.

FT: SHIFTED COSINE



FREQUENCY-SHIFTING: EXAMPLE



CONVOLUTION

An important operation in astronomy.

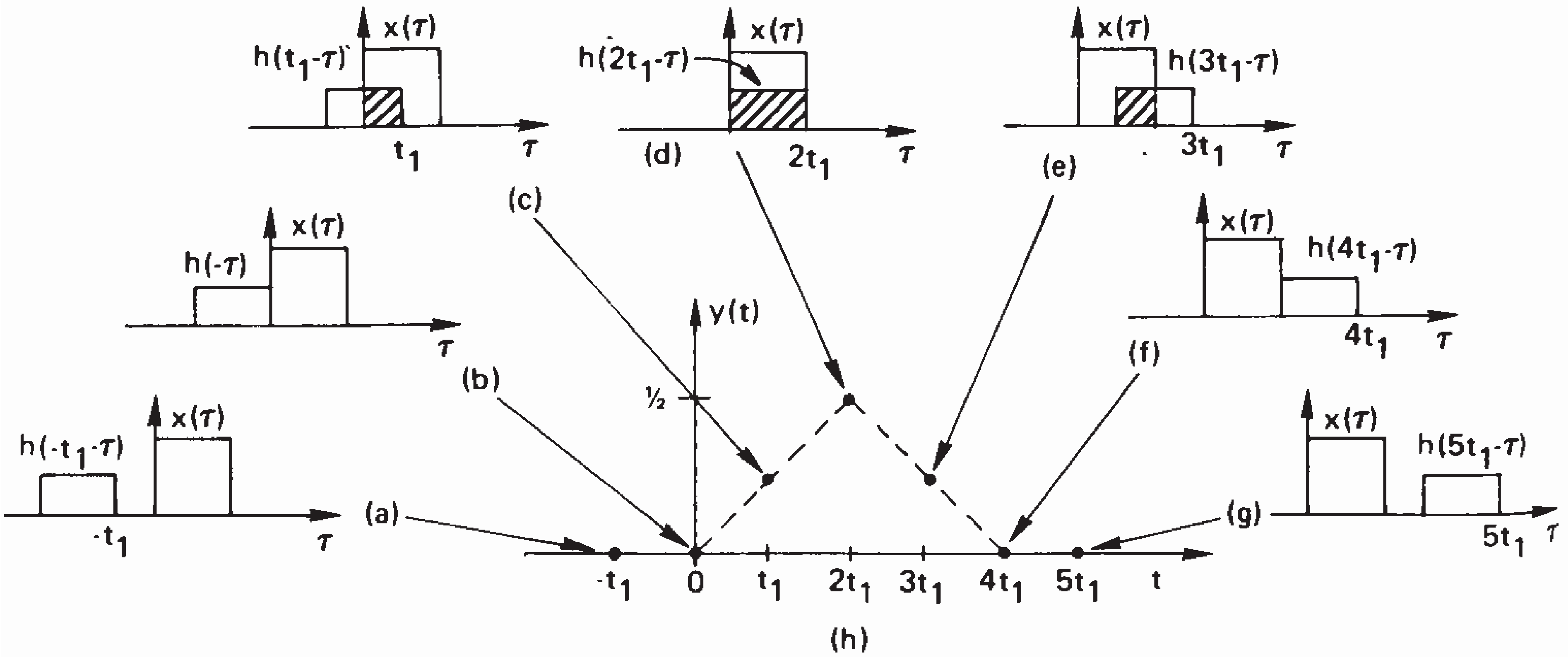
Observational data is the real data modified by the response of the detectors.

Convolution is used in smoothing operations.

Convolution is also used in filters

CONVOLUTION OF TWO FUNCTIONS:

$$f(t) \odot h(t) = \int_{-\infty}^{+\infty} f(\tau)h(t-\tau)d\tau$$



PROPERTIES OF FT: CONVOLUTION THEOREM

FT of a convolution is the product of two FTs. Convolution of $g_1(t)$ and $g_2(t)$ is defined as,

$$g_1(t) \odot g_2(t) = \int_{-\infty}^{+\infty} g_1(\tau) g_2(t-\tau) d\tau$$

If $G_1(\omega)$ and $G_2(\omega)$ are FT of g_1 and g_2 respectively then the statement of convolution theorem is,

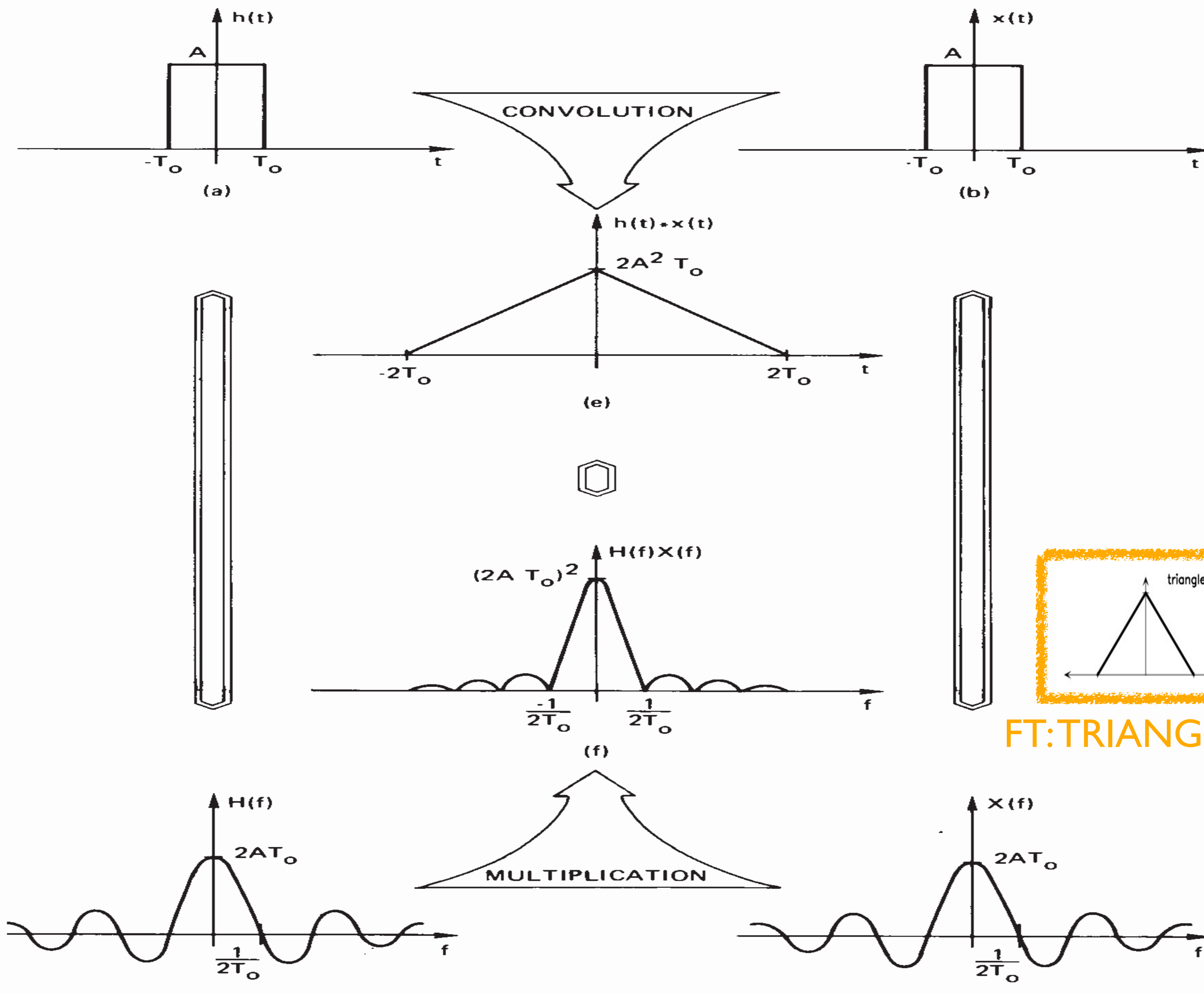
$$\mathbf{F}\left[G_1(\omega) G_2(\omega) \right] = \int_{-\infty}^{+\infty} g_1(\tau) g_2(t-\tau) d\tau$$

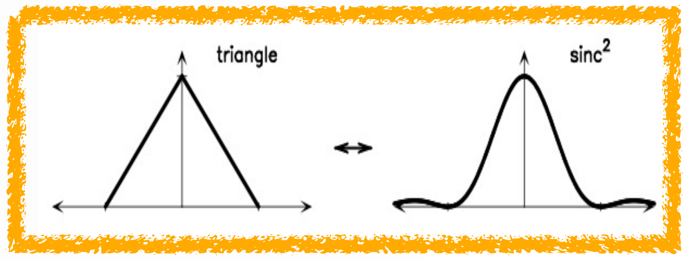
We shall call, $G_1(\omega)G_2(\omega) = H(\omega)$ and $H(\omega)$ is the FT of $h(t)$.

Properties of FT: Convolution Theorem

$$
\begin{aligned}
h(t) &= \int_{-\infty}^{+\infty} H(\omega) e^{+i\omega t} d\omega = \int_{-\infty}^{+\infty} G_1(\omega) G_2(\omega) e^{+i\omega t} d\omega \\
&= \int_{-\infty}^{+\infty} G_1(\omega) \int_{-\infty}^{+\infty} [g_2(t') e^{-i\omega t'} dt'] e^{+i\omega t} d\omega \\
&= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} G_1(\omega) g_2(t') e^{i\omega(t-t')} dt' d\omega \\
&= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} G_1(\omega) g_2(t-\tau) e^{i\omega\tau} d\tau d\omega \\
&= \int_{-\infty}^{+\infty} g_2(t-\tau) \left[\int_{-\infty}^{+\infty} G_1(\omega) e^{i\omega\tau} d\omega \right] d\tau = \int_{-\infty}^{+\infty} g_2(t-\tau) g_1(\tau) d\tau
\end{aligned}
$$

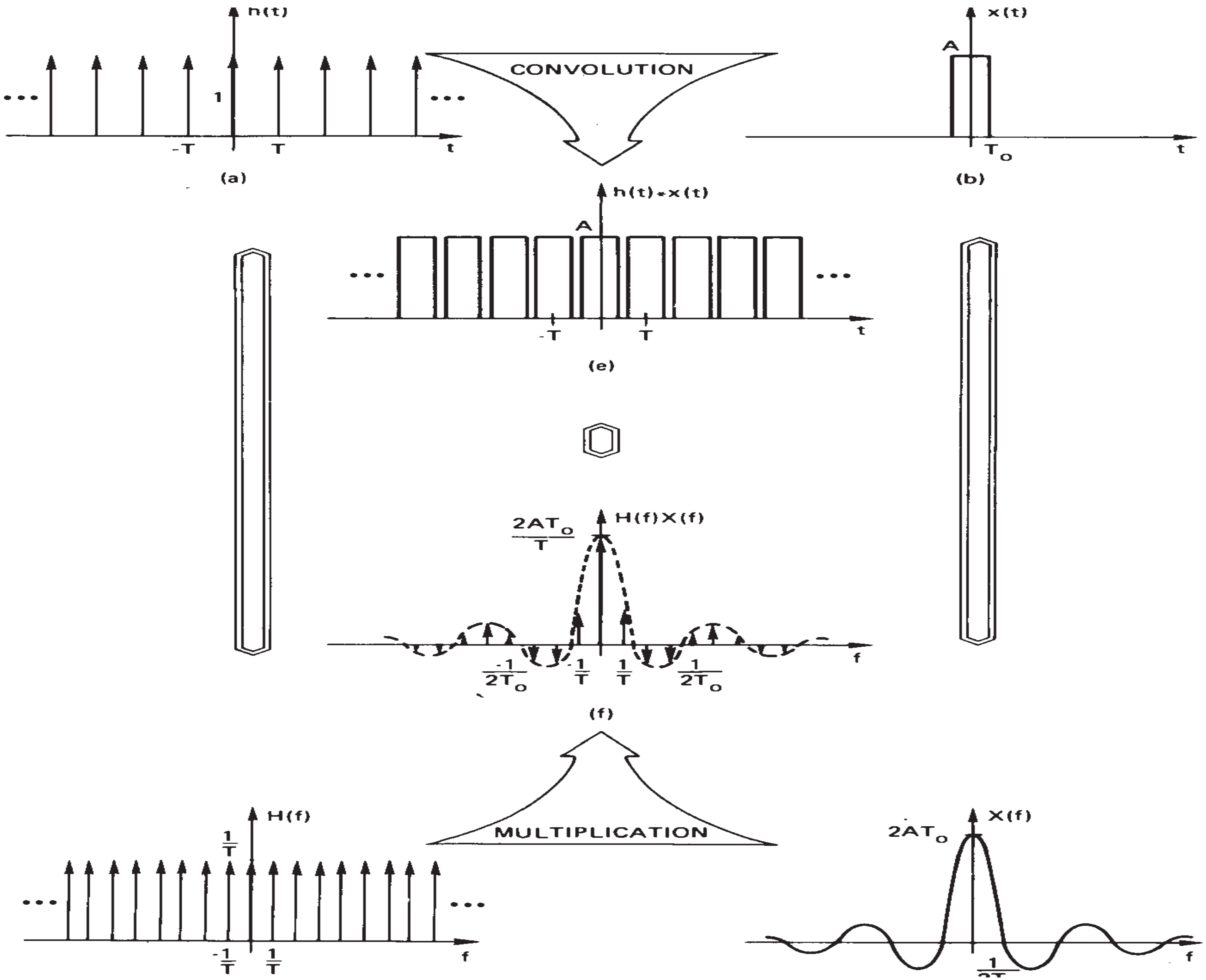
APPLICATION OF CONVOLUTION THEOREM:



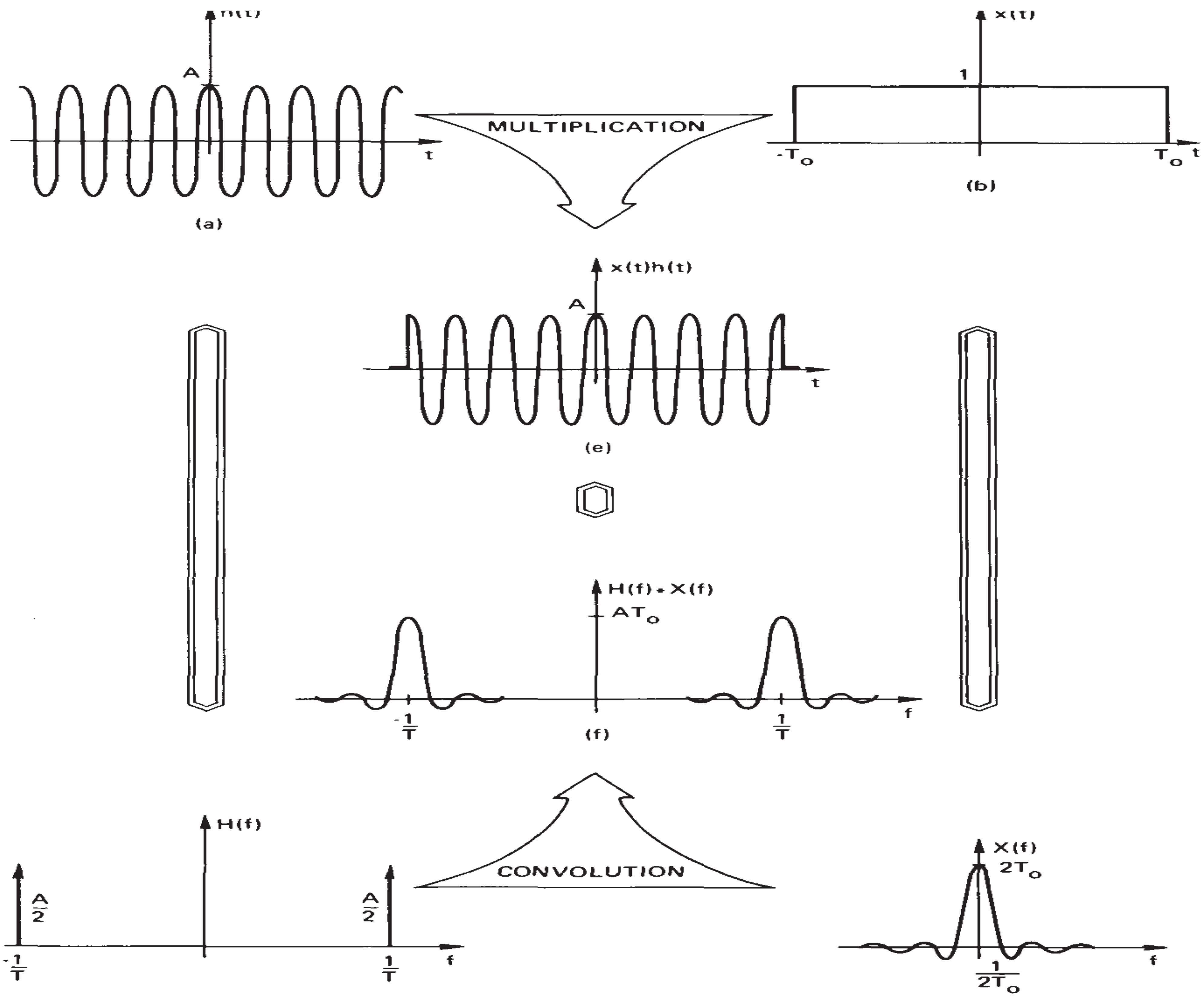


FT: TRIANGLE FUNCTION

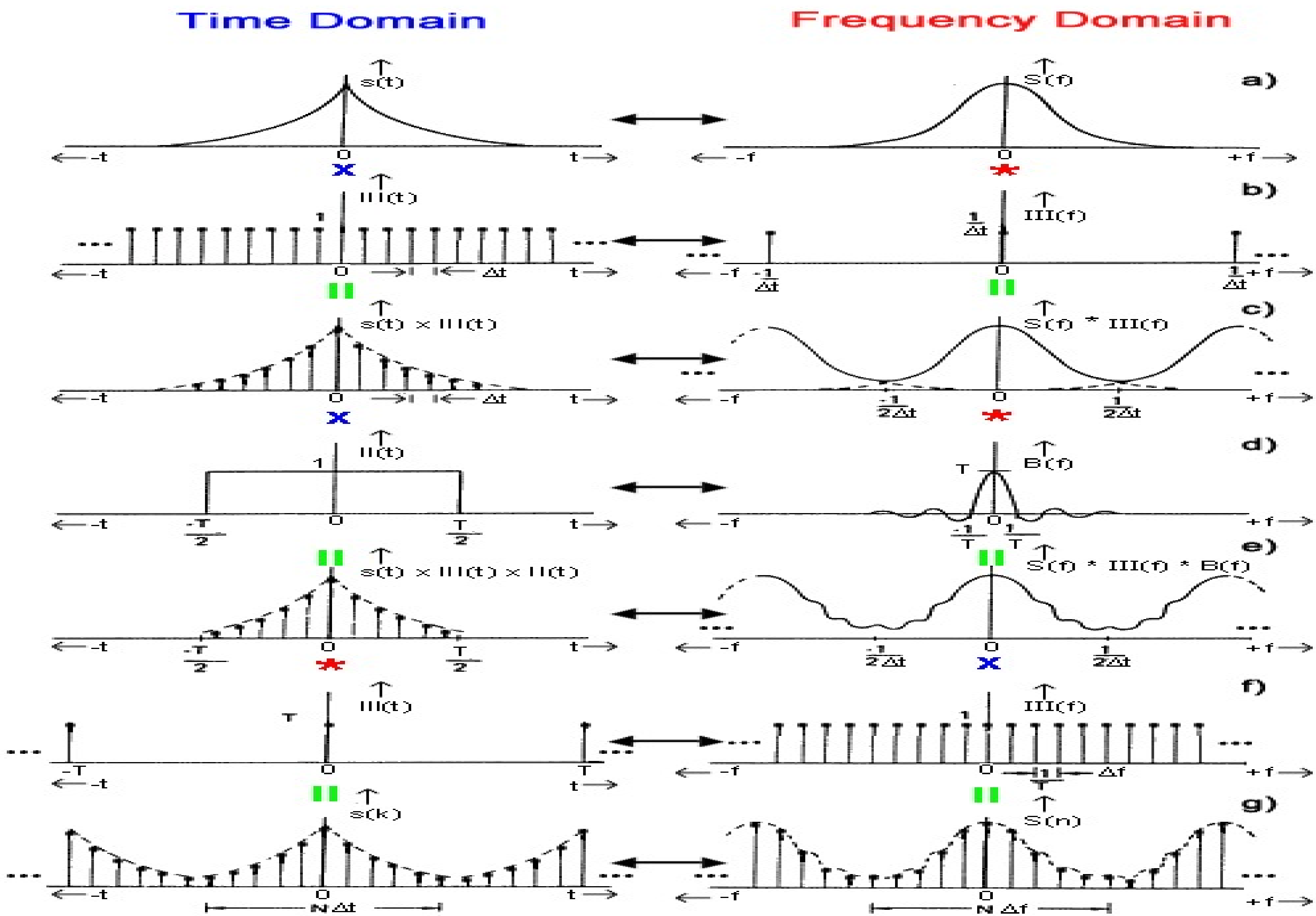
APPLICATIONS CONVOLUTION THEOREM:



Application of convolution Theorem:

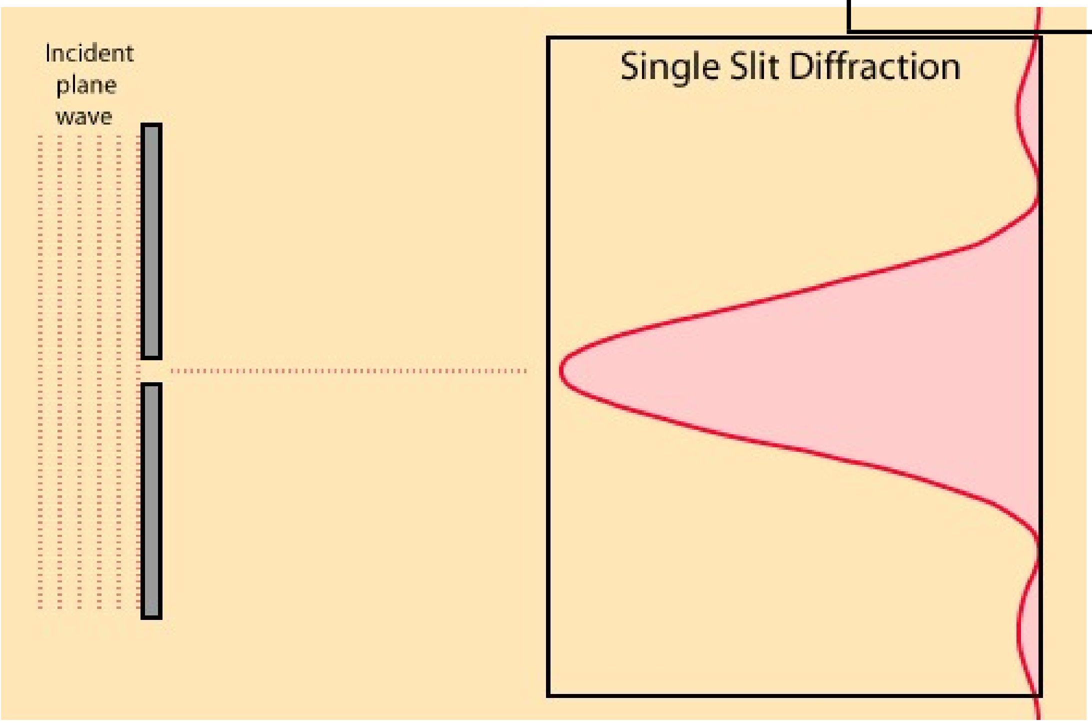


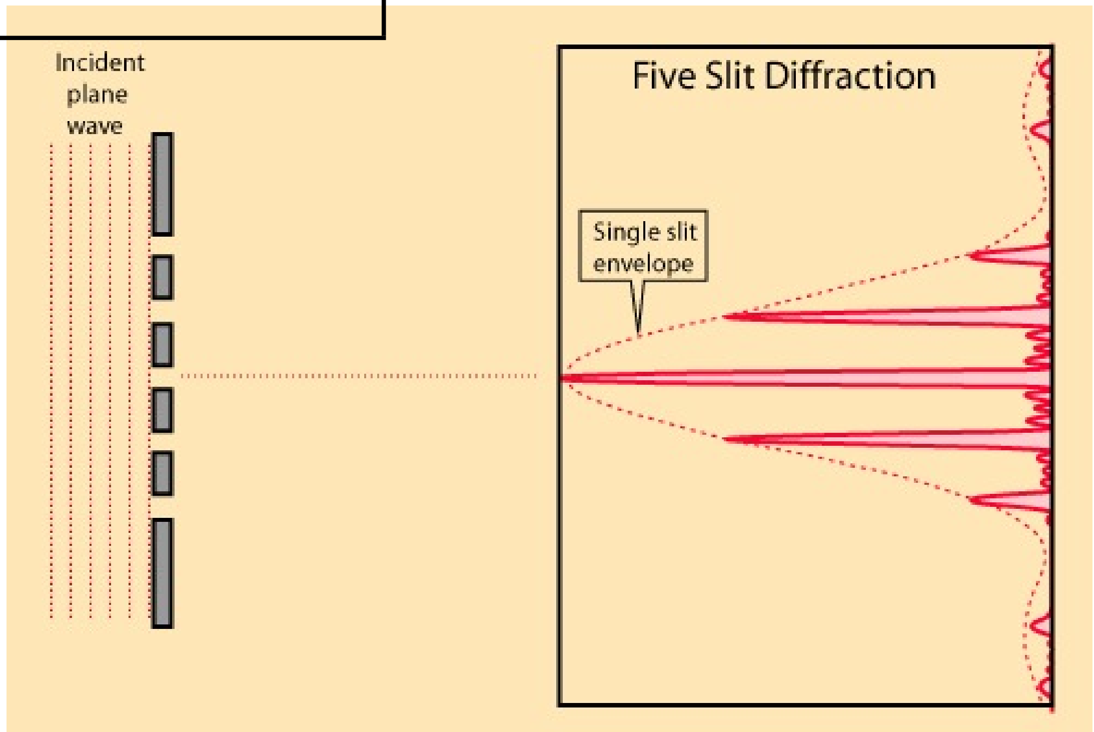
APPLICATION OF CONVOLUTION THEOREM:

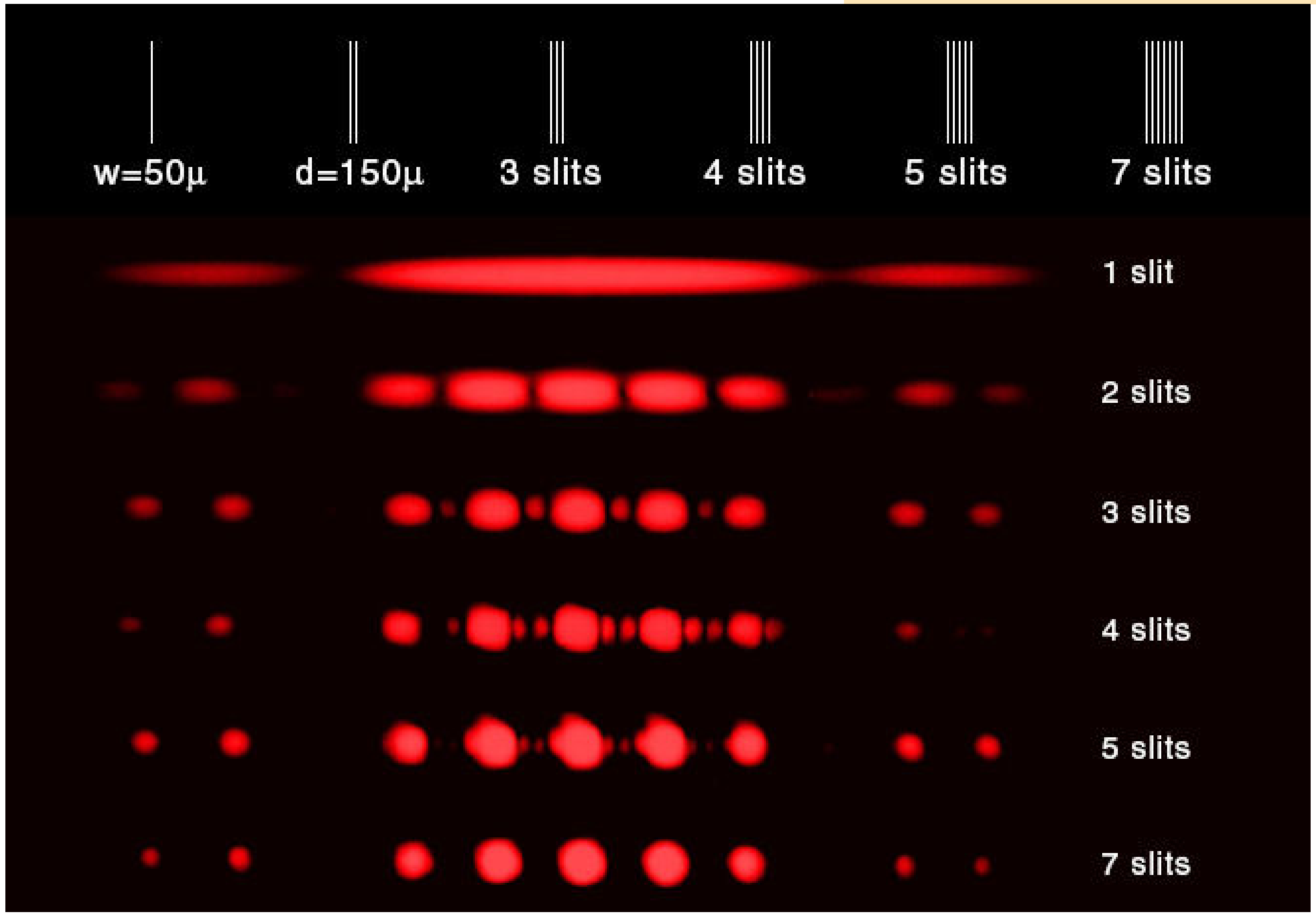


Example: Write out FT of M(x) * [B(x) o I(x) + N(x)]

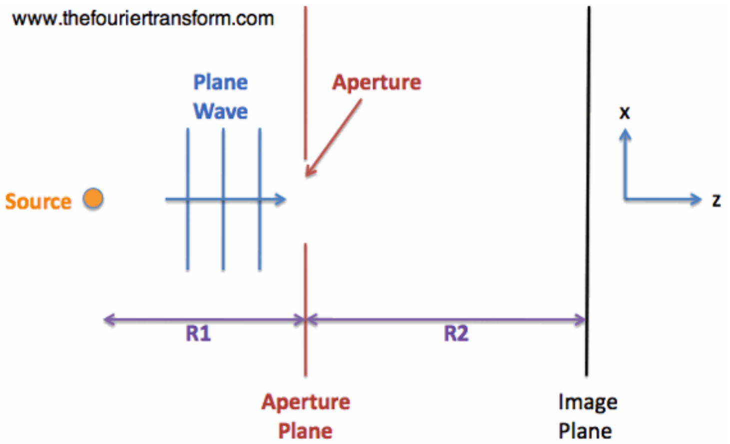
Optics example:







EM fields on the image plane = Fourier transform of the aperture function



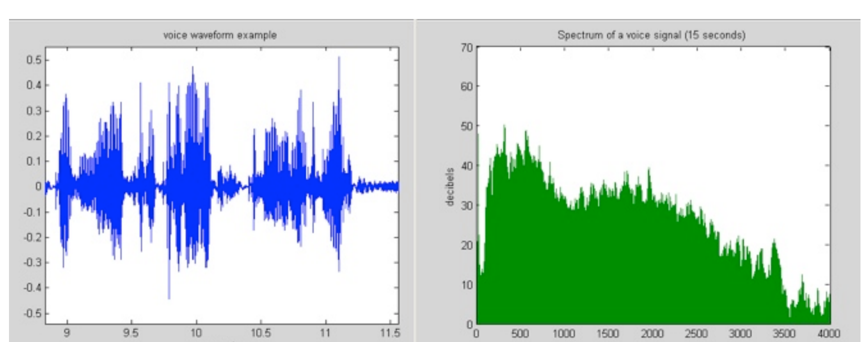
Power spectrum and Parseval's theorem

Power spectrum of a signal is defined by the modulus square of the FT. Any function and its FT obey the condition that

$$\int_{-\infty}^{+\infty} |f(t)|^2 dt = \int_{-\infty}^{+\infty} |F(\omega)|^2 d\omega \qquad (12)$$

This is known “Rayleigh theorem” or “Parseval’s theorem”.

$$\begin{aligned}
\int_{-\infty}^{+\infty} |f(t)|^2 dt &= \int_{-\infty}^{+\infty} |f(\omega)|^2 d\omega = \int_{-\infty}^{+\infty} f^*(t) f(t) dt \\
&= \int_{-\infty}^{+\infty} \left[\int_{\infty}^{+\infty} F(w) e^{i\omega t} d\omega \right]^* f(t) dt = \int_{-\infty}^{+\infty} \left[\int_{\infty}^{+\infty} F^*(w) e^{-i\omega t} d\omega \right] f(t) dt \\
&= \int_{-\infty}^{+\infty} F^*(w) \left[\int_{\infty}^{+\infty} f(t) e^{-i\omega t} dt \right] d\omega = \int_{-\infty}^{+\infty} |f(\omega)|^2 d\omega
\end{aligned}$$



CORRELATIONS:

Correlation of two functions,

$$g_1(t) \star g_2(t) = \int_{-\infty}^{+\infty} g_1(t) g_2(t+\tau) d\tau$$

When g_2 is an even function the correlation becomes identical to convolution.

When $g_1 = g_2$ it is called "auto-correlation".

Cross-correlation theorem,

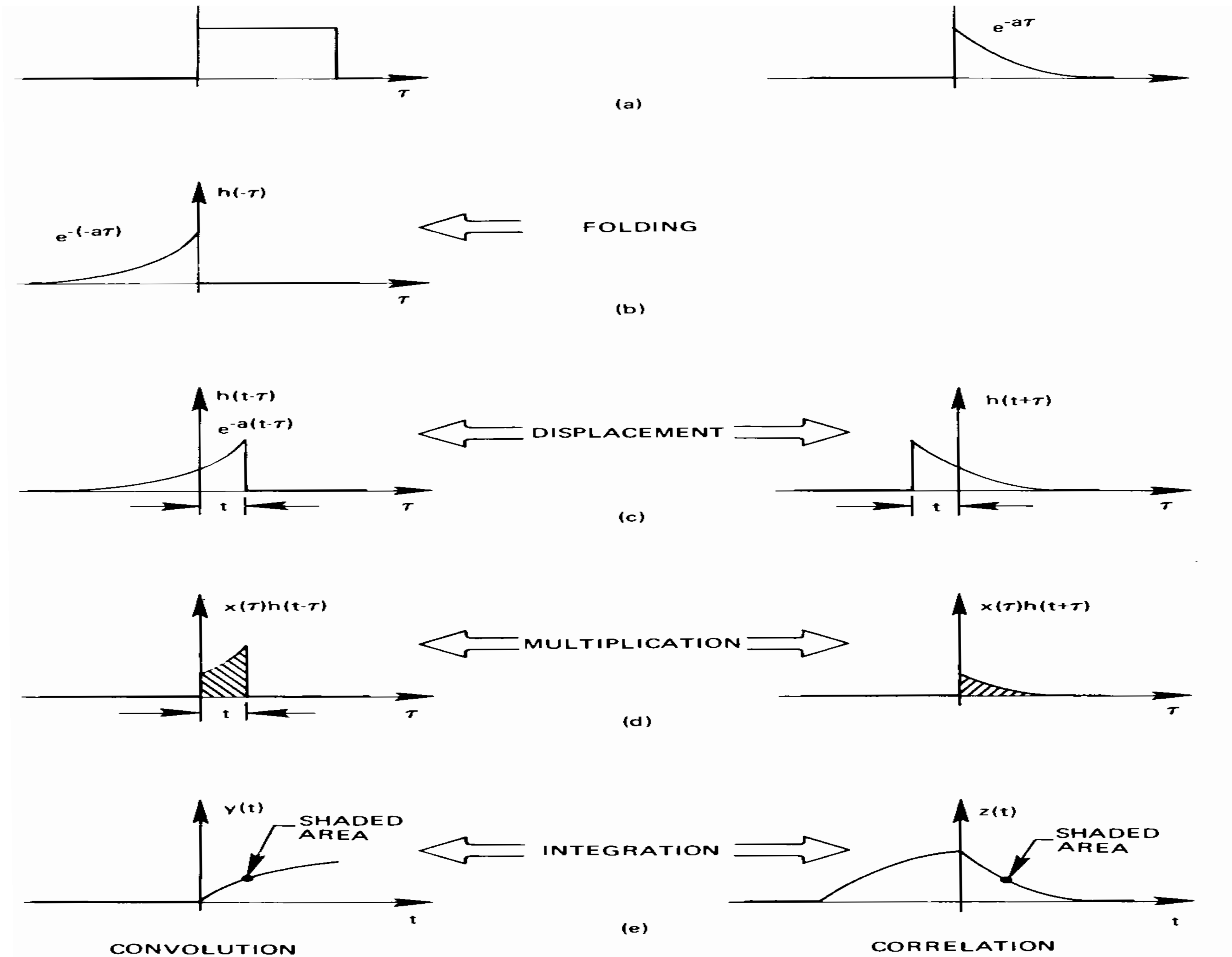
$$\mathbf{F}\left[G_1(\omega) G_2^*(\omega)\right] = \int_{-\infty}^{+\infty} g_1(\tau) g_2(t+\tau) d\tau$$

The corresponding auto-correlation theorem is known as Wiener-Khinchin theorem,

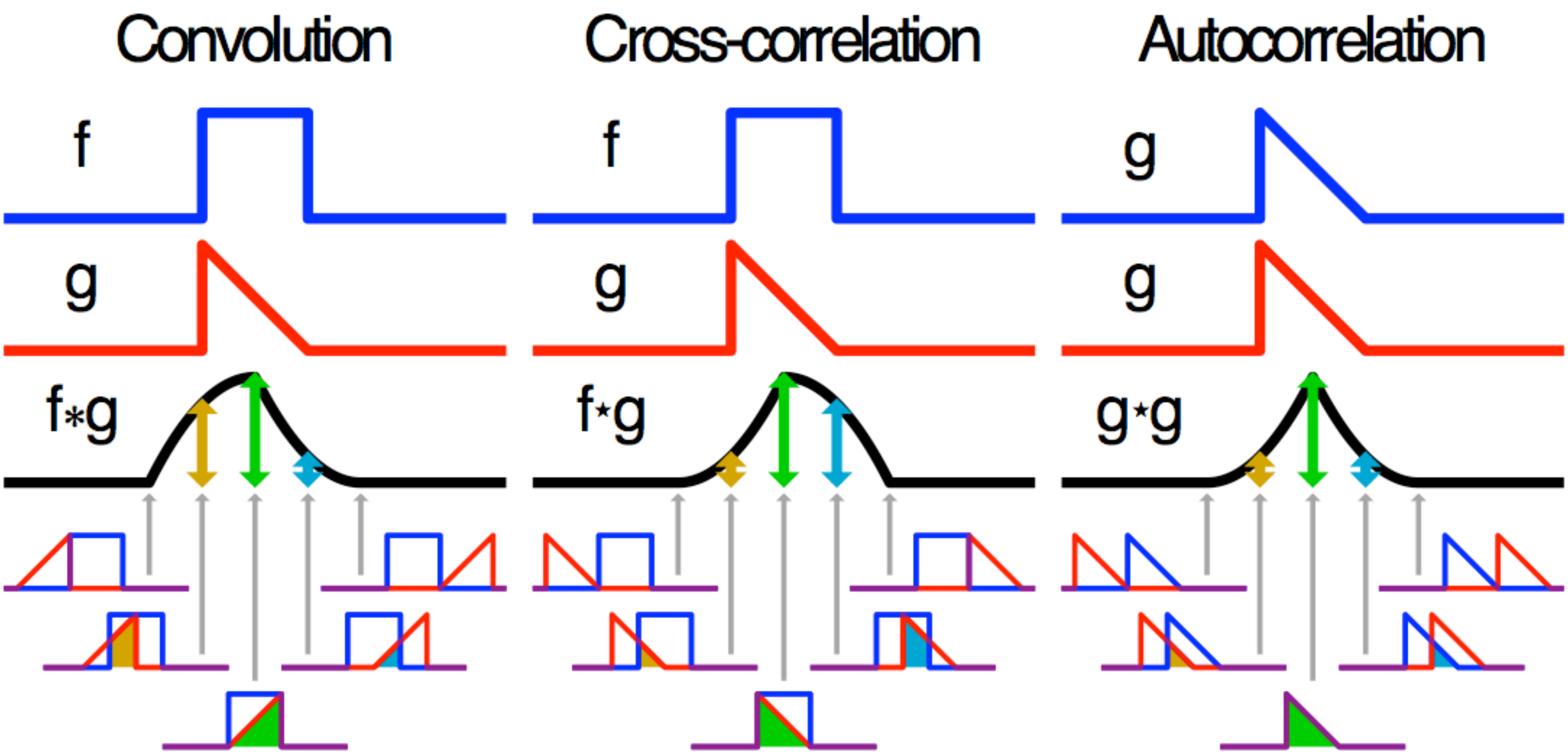
$$(g \star g) = |G|^2$$

In words, autocorrelation function is the power spectrum, or equivalently, the autocorrelation is the inverse Fourier transform of the power spectrum. Many radio-astronomy instruments compute power spectra using autocorrelations and this theorem.

COMPARISION OF CONVOLUTION & CORRELATION



COMPARISION OF CONVOLUTION & CORRELATION



END - PART 1